NEW CHARACTERIZATIONS OF EULERIAN AND BIPARTITE BINARY MATROIDS

M. M. SHIKARE

Department of Mathematics, University of Pune, Pune 411 007, India
(E-mail: mms@math.unipune.ernet.in)

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We show that binary matroid is Eulerian if and only if every element of it is contained in an odd number of circuits. It is proved that a binary matroid is Eulerian if and only if the ground set has an odd number of partitions into circuits. Corresponding results for bipartite binary matroids are derived.

Key Words : Binary Matroid; Eulerian Matroids; Bipartite Matroids; Circuit; Cutset; Partitions

1. INTRODUCTION

A matroid $M$ is a pair $(S, \mathcal{F})$ where $S$ is a finite set and $\mathcal{F}$ is a collection of subsets of $S$, called independent sets of $M$, with the following properties:

1. $\phi \in \mathcal{F}$,
2. if $X \in \mathcal{F}$ and $Y \subseteq X$ then, $Y \in \mathcal{F}$,
3. if $X \in \mathcal{F}$ and $Y \in \mathcal{F}$ and $|X| > |Y|$ then there exists an element $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

The set $S$ is known as ground set of $M$. A maximal independent set of $M$ is a base of $M$. A subset of $S$ not belonging to $\mathcal{F}$ is said to be dependent. A minimal dependent subset of $S$ is called a circuit of $M$. The set of all bases of $M$ will be denoted by $\mathcal{B}$. The matroid $M^*$ whose ground set is $S$ and whose set of bases is $\mathcal{B}^* = \{S \setminus B : B \in \mathcal{B}\}$ is called the dual of a matroid $M$. A circuit of $M^*$ is called a cutset (or co-circuit) of $M$.

Generalizing graph-theoretic concepts we call a matroid $M = (S, \mathcal{F})$ an Eulerian matroid if there exist disjoint circuits $C_1, C_2, \ldots, C_n$ such that

$$S = C_1 \cup C_2 \cup \ldots \cup C_n.$$ 

We define $M$ to be a bipartite matroid if every circuit of $M$ has even number of elements.

A matroid $M = (S, \mathcal{F})$ is defined to be binary if the symmetric difference of any set of circuits of $M$ is a union of disjoint circuits of $M$. The dual $M^*$ of a binary matroid $M$ is also a binary matroid (see [3]). We have the alternative definition that a matroid is binary if and only if
for every circuit \( C \) and a cutset \( D \) of \( M \) \( \mid C \cap D \mid \) is even. If \( M = (S, \mathcal{F}) \) is a matroid and \( T \subseteq S \) then the deletion of \( T \) from \( M \) denoted by \( M_{/T} \) is a matroid \( (S - T, \mathcal{F}') \) where a subset \( X \) of \( S - T \) is in \( \mathcal{F}' \) if and only if \( X \in \mathcal{F} \). We follow the notation and terminology of [3] and [7].

We need the following results.

Proposition 1.1\(^2\) — Let \( M = (S, \mathcal{F}) \) be a binary matroid and \( X \subseteq S \). Then \( X \) is disjoint union of circuits of \( M \) if and only if \( X \) intersects each cutset evenly.

Result 1.2\(^6\) — A binary matroid is Eulerian if and only if its dual matroid is bipartite.

2. MAIN RESULTS

Theorem 2.1 — A binary matroid \( M = (S, \mathcal{F}) \) is Eulerian if and only if every element of \( S \) is contained in an odd number of circuits of \( M \).

This result is not true for non-binary matroids as shown by the following example.

Example — Consider the uniform matroid \( U_{6, 2} \) of rank 2 on a six element set. This is a non-binary Eulerian matroid. Every 3-element set is a circuit. The number of circuits containing an element \( x \) is equal to the number of ways to choose remaining two elements in a circuit from the remaining 5-elements of \( U_{6, 2} \). Thus, every element of \( U_{6, 2} \) is contained in ten circuits of \( U_{6, 2} \). On the other hand, \( U_{4, 2} \) is non-binary matroid. Every 3-element set in it is a circuit. So number of circuits containing an arbitrary element \( x \) is equal to the number of ways to choose remaining 2 elements in a circuit from the remaining 3-elements of \( U_{4, 2} \). This number is 3, an odd number, but \( U_{4, 2} \) is not Eulerian.

PROOF OF THE THEOREM: Suppose that a binary matroid \( M = (S, \mathcal{F}) \) is Eulerian. So by result 1.2, every cutset of \( M \) has even cardinality. Let \( x \in S \). If \( x \) is a loop then it belongs to exactly one cycle i.e., to an odd number of cycles and the result is proved. Assume therefore that \( x \) is not a loop. We show that \( x \) is contained in an odd number of circuits of \( M \). Let \( \Delta \mathcal{F}_x \) denote the set of circuits containing \( x \) and \( \Delta \mathcal{F}_x \) denote the symmetric difference of members of \( \mathcal{F}_x \). Note that \( \Delta \mathcal{F}_x \) consists of those element which are contained in an odd number of circuits containing \( x \). Let \( C(x_1, x_2, \ldots, x_n) \) denote the number of circuits containing \( x_1, x_2, \ldots, x_n \). Firstly, we prove that \( X = \Delta \mathcal{F}_x \cup \{x\} \) intersects each cutset of \( M \) in an even number of elements. Let \( D \) be a cutset of \( M \). If \( x \notin D \), then \( D \cap X = D \cap (\Delta \mathcal{F}_x \cup \{x\}) = D \cap \Delta \mathcal{F}_x \) is even by binarity of \( M \). Now, let \( x \in D \) and \( D \cap X \) has odd number of elements. Then we will arrive at a contradiction. Suppose \( D = \{x_1, x_2, \ldots, x_n\} \). So by Result 1.2, \( n \) is odd. The set \( (D \cap X) - x \) is even and therefore the sum

\[
\sum_{i=1}^{n} C(x, x_i) \text{ is also even. So, at least one of the } n \text{ terms must be even. Let } C(x, x_i) \text{ be even and let } D_2 \text{ be any cutset containing } x \text{ but not } x_1. \text{ Let } C(x, x_1) \text{ be even and let } D_2 \text{ be any cutset containing } x \text{ but not } x_1. \text{ Let } D_2 = \{x, y_1, y_2, \ldots, y_{n'}\}. \text{ By Result 1.2, } n' \text{ is odd. Now } C(x, x_1) = \sum_{i=i}^{n'} C(x, x_1, y_i) \text{ is even, therefore at least one of the } n' \text{ terms must be even; say } C(x, x_1, y_1) \text{ is even.}
\]
Let $D_3$ be any cutset containing $x$ but neither $x_1$ nor $y_1$ etc. Continue this process until cutset $D_i$ can not be chosen. Then the $i$ elements $x_1, y_1, \ldots$ together with $x$ form a circuit and so $C(x, x_1, y_1, \ldots) = 1$. But we also have $C(x, x_1, y_1, \ldots)$ even, which is a contradiction. Therefore, $D \cap X$ must have an even number of elements.

Now by Proposition 1.1, $X$ is a disjoint union of circuits of $M$. Also, $\Delta \lambda_x$ is a disjoint union of circuits of $M$ and $X = \lambda_x \bigcup \{x\}$. So, we must have $x \in \Delta \lambda_x$. Consequently, the number of circuits containing $x$ must be odd.

Conversely, suppose that a matroid $M = (S, \mathcal{F})$ is binary and every element is contained in an odd number of circuits. In order to prove that $M$ is Eulerian, we prove that every cutset of $M$ is of even size.

Let $D$ be any cutset of $M$. As mentioned earlier, $C(x)$ denotes the number of circuits containing the element $x$. Since $M$ is binary every circuit of $M$ intersects $D$ evenly the sum $\Sigma C(x)$ over all $x \in D$ will count each circuit an even number of times. So, the sum $\sum_{x \in D} C(x)$ will be even. By assumption each term is odd, so there must be an even number of terms, thereby showing $D$ to be even. Consequently, $M$ is Eulerian.

As an immediate consequence of the above theorem, we have the following characterization of Eulerian binary matroid.

**Corollary 2.2 —** A binary matroid $M = (S, \mathcal{F})$ is Eulerian if and only if symmetric difference of all circuits of $M$ equals $S$.

Combining Theorem 2.1 with the Result 1.2, we get a corresponding characterisation for bipartite matroids.

**Corollary 2.3 —** A binary matroid $M = (S, \mathcal{F})$ is bipartite if and only if every element is contained in an odd number of cutsets of $M$.

The following theorem gives a characterisation of binary Eulerian matroids in terms of the number of partitions of a ground set into circuits of a matroid. This generalizes to the binary matroids the characterization of Eulerian graphs due to Bondy and Halberstan

**Theorem 2.4 —** A binary matroid $M = (S, \mathcal{F})$ is Eulerian if and only if $S$ can be partitioned into circuits of $M$ in odd number of ways.

**Proof**: If in a binary matroid $M$ the ground set $S$ can be partitioned into circuits of $M$ in odd number of ways then it surely has at least one circuit partition, therefore $M$ is Eulerian.

Now suppose that $M = (S, \mathcal{F})$ is Eulerian and binary. Let $x \in S$ and $C_1, C_2, \ldots, C_k$ be the circuits containing $x$. Then by Theorem 2.1, $k \equiv 1 \pmod{2}$. We proceed by induction on $|S|$. If $|S| = 1$ then $S$ has a trivial partition consisting of a loop.

Let $|S| = n > 1$. If $S_i = S - C_i = \phi$ for some $i$, then $k = 1$ and $S$ is a circuit with $S = \{C_1\}$ as its unique circuit partition. Assume therefore that $S_i \neq \phi, 1 \leq i \leq k$. By induction the ground set $S - C_i$ of the matroid $M - C_i = (S - C_i, \mathcal{F} - C_i)$ has an odd number of circuit partitions. This yields an odd number of circuit partitions of $S$ in $M$; containing the circuit $C_i$. Denote this number by $\tau(C_i)$ and denote by $\tau(S)$ the number of all circuit partitions of $S$ in $M$.

Consequently,
\[ \tau(S) = \sum_{i=1}^{k} \tau(C_i) = k \equiv 1 \pmod{2}, \]

i.e., \[ \tau(S) \equiv 1 \pmod{2}. \]

**Remark 2.5:** This also does not hold for non-binary matroids as shown by the following example.

Consider the uniform matroid \( U_{6,2} \) of the above example. Since every 3-element set is a circuit, there are \( ^6C_3 \), i.e., 20 circuits. Now any circuit and its complement in the 6-element set which is also a circuit form a circuit partition of the ground set of \( U_{6,2} \). Thus, in all there are ten, an even number of circuit partitions of the ground set of \( U_{6,2} \).

Combining Theorem 2.4 and the Result 1.2, we give another characterization of binary bipartite matroids in terms of the number of partitions of ground set into cutsets of a matroid.

**Corollary 2.6** — A binary matroid \( M = (S, \mathcal{F}) \) is bipartite if and only if \( S \) can be partitioned into cutsets of \( M \) in odd number of ways.

### 3. Algorithm for Circuit Partitions of a Ground Set of an Eulerian Matroid

Here, we present a description of an algorithm for the construction of the set \( \mathcal{P}(S) \), of all circuit partitions of \( S \) in a binary Eulerian matroid \( M = (S, \mathcal{F}) \). Every Eulerian matroid has at least one circuit, \( C \) (say) and \( M \setminus C = (S-C, \mathcal{F}') \) is also Eulerian. We use this to describe an algorithm to obtain circuit partitions of arbitrary binary Eulerian matroid.

Suppose \( M = (S, \mathcal{F}) \) is a Eulerian matroid and \( S = \{x_1, x_2, \ldots, x_n\} \). Choose an arbitrary element \( x \in S \), \( x = x_1 \) say. A first subroutine produces in lexicographic order the set \( \mathcal{L}_x \) of all circuits containing \( x \). For \( C_i \in \mathcal{L}_x \), the matroid \( M \setminus C_i = (S-C_i, \mathcal{F}') \) is Eulerian. A second subroutine produces the set \( \mathcal{P}_i \) of all circuit partitions of \( S \) in the matroid \( M \) which have circuit \( C_i \) in common. This is achieved by determining \( \mathcal{P}(S-C_i) \) and forming

\[ \mathcal{P}_i = \{\{C_i\} \cup P | P \in \mathcal{P}(S-C_i)\}. \]

Then we note that \( \mathcal{P}_i \cup \mathcal{P}_j = \phi \) for \( i \neq j \), \( 1 \leq i, j \leq |\mathcal{L}_x| = \gamma_x \) where \( \gamma_x \) is the number of circuits containing \( x \).

Now, it follows that

\[ \gamma_x \]

\[ \mathcal{P}(S) = \bigcup_{i=1}^{\gamma_x} \mathcal{P}_i \]

is the required set. Here each \( \mathcal{P}_i \) is a partition of \( S \) into circuits of \( M \).
Remark 3.1: (1) We have $|\mathcal{P}(S)| = 1$ for a matroid $M$ each of whose component is a circuit.

(2) By Theorem 2.4, the number of partitions of $S$ into circuits of $M$ is odd if and only if $M = (S, \mathcal{F})$ is binary Eulerian matroid. Hence,

$$|\mathcal{P}(S)| \equiv 1 \pmod{2}$$

in case of binary Eulerian matroid.

REFERENCES

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